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ANALYSIS OF AIR-BUBBLE PLUMES

Klas Cederwall, et al

California Institute of Technology

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Klas Cederwall

and

John D. Ditmars

W. M. Keck Laboratory of Hydraulics and Water Resources

Division of Engineering and Applied Science

CALIFORNIA INSTITUTE OF TECHNOLOGY

Pasadena, California

Report No. KH-R-24

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by

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ABSTRACT

The air-bubble plume induced by the steady release of air into water has been analyzed with an integral technique based on the equations for conservation of mass, momentum and buoyancy. This approach has been widely used to study the behavior of submerged turbulent jets and plumes. The case of air-bubble induced flow, however, includes additional features. In this study the compressibility of the air and the differential velocity between the rising air bubbles and the water are introduced as basic properties of the air-bubble plume in addition to a fundamental coefficient of entrainment and a turbulent Schmidt number characterizing the lateral spreading of the air bubbles.

Theoretical solutions for two- and three-dimensional air-bubble systems in homogeneous, stagnant water are presented in both dimensional and normalized form and compared to existing experimental data. The further complication of a stratified environment is briefly discussed since this case is of great practical interest.

This paper is to be considered as a progress report, as future experimental verification of various hypotheses is needed.

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NOTATIONS

The main symbols used in this study are listed below. The SI-system is used consistently throughout this study.

y, r	lateral coordinates
x	vertical coordinate
Δx	virtual displacement of source
z	normalized coordinate
b	nominal half-width of plume
c	local concentration of air by volume
c_m	centerline value of c
u	local plume velocity
u_m	centerline value of u
u_b	slip velocity of air bubbles
H	total depth
H_o	atmospheric pressure height
h	depth of surface layer of horizontal flow
Q	volume flux
q	rate of air flow
q_o	q at the surface
M	momentum flux
B	buoyancy flux
ρ_a	density of ambient water
ρ_o	ρ_a at source level in case of stratified water
ρ_w	local density of water within plume

NOTATIONS (continued)

$\Delta\rho_w$	$\max \{\rho_w - \rho_a\}$
ρ_m	local density of air-water mixture within plume
ρ_{air}	density of air
α	coefficient of entrainment
λ	spreading ratio
S, G, P	plume parameters as defined in the text

Natural water bodies such as lakes and impoundments are usually stratified. The thermal (density) stratification is a factor which effectively controls convection and diffusion in the lake. The most stable stratification conditions exist during the summer months when the lake or reservoir exhibits large thermal gradients. As a result the water quality of the lower layers may be reduced significantly since the dissolved oxygen content is no longer replenished by surface reaeration. Artificial mixing may then be an effective way of improving water quality of the water body. There are several methods of accomplishing destratification; the use of air-curtain systems seems to be a good alternative to mechanical pumping. However, little is known about the air-bubble induced circulation in stratified waters. The final part of this report will discuss this problem.

THE FLOW MODEL

The interaction between the rising bubbles and the ambient water is a complex function depending on several parameters. The idealized model we will use is based on the following selection of significant variables:

q_o the volume rate of air flow at atmospheric pressure (in m^3/s for point source and in m^2/s for line source)

u_b the differential velocity of the air bubbles relative to the water (slip velocity)

$\frac{\rho_a - \rho_{air}}{\rho_a} g$ apparent gravity where ρ_a and ρ_{air} are the density of the ambient water and the air respectively and g the acceleration of gravity

H the depth above the air flow source

H_o piezometric head equivalent to the atmospheric pressure (H_o is regarded as a constant ≈ 10.4 m)

u_b, the terminal rise velocity of an air-bubble in stagnant, ambient water, is assumed to be attained near the source and then remain constant throughout the depth and to be equal for all bubbles. This has been found to be a good approximation despite the fact that the bubbles probably vary in size in the plume. *

The expansion of the air bubbles as they rise through the water is neither adiabatic nor truly isothermal but an intermediate process. We will here for simplicity assume isothermal conditions – the

---

* For example, Haberman and Morton show that in the range of diameters 10^{-3} to 10^{-2} m, u_b changes only from about 0.20 to 0.30 m/s, see discussion in a later section of this paper.

When the air-bubbles rise through the ambient water, the induced vertical flow is likely to resemble the process of turbulent diffusion from a submerged source of buoyancy as indicated originally by Taylor (1955). This similarity with the plume case is the basis for the present work. Hence, to solve the problem we will use the integral technique first proposed by Morton, Taylor and Turner (1956). The study is restricted to the zone where the horizontal flow near the water surface does not affect the vertical flow, that is for $x < H-h$, see Fig. 1. It has been found empirically that h is approximately 0.25H in the two-dimensional case and somewhat less for a point-source.

1A. A Point Source in Homogeneous Ambient Fluid

The rate of entrainment at the edge of the plume is assumed to be proportional to the mean axial plume velocity. Furthermore, we assume that the lateral profiles of velocity and density deficiency are similar at all heights and that they can be approximated by Gaussian distributions. Hence, for the velocity

$$u = u_m \cdot e^{-r^2/b^2} \tag{3}$$

where u is a local mean velocity and u_m the centerline velocity. b is the nominal half-width of the plume related to the standard deviation of the velocity distribution by

$$2\sigma = \sqrt{2}\, b \tag{4}$$

The density distribution is given by

$$(\rho_a - \rho_m) = \Delta\rho_m \cdot e^{-r^2/(\lambda b)^2} \tag{5}$$

and hence we arrive at the following buoyancy flux relation

$$\pi u_m \Delta\rho_m \frac{\lambda^2 b^2}{1+\lambda^2} + \pi u_b \Delta\rho_m \lambda^2 b^2 = q_o \rho_a \frac{H_o}{H_o+H-x} \tag{12}$$

Similarly, for the momentum flux with $\rho_m \approx \rho_a$ according to Boussinesq assumption

$$M = \int_0^\infty 2\pi u^2 \rho_a r\, dr = \frac{\pi u_m^2 \rho_a b^2}{2} \tag{13}$$

The driving force of the plume is the buoyancy, and the momentum flux equation is

$$\frac{dM}{dx} = \int_0^\infty 2\pi (\rho_a - \rho_m) g\, r\, dr = \pi g \Delta\rho_m \lambda^2 b^2 \tag{14}$$

which combined with Eq. (13) yields

$$\frac{d(u_m^2 b^2)}{dx} = 2g \frac{\Delta\rho_m}{\rho_a} \lambda^2 b^2 \tag{15}$$

Substituting $\frac{\Delta\rho_m}{\rho_a}$ from Eq. (15) into Eq. (12) gives us two equations to solve

$$\frac{d(u_m b^2)}{dx} = 2\alpha u_m b \tag{16}$$

$$\frac{d(u_m^2 b^2)}{dx} = \frac{2 g q_o H_o}{\pi(H_o+H-x)\left(\frac{u_m}{1+\lambda^2} + u_b\right)} \tag{17}$$

Eqs. (16) and (17) must be solved to obtain the centerline values u_m and b as a function of x and $\Delta\rho_m$ is then found from Eq. (12). Since a solution to the differential equations cannot be obtained in closed analytical form, a numerical integration has to be carried out. The numerical solution follows from a direct step-by-step integration of

air-bubble plumes and for a wide range of q_o.* Hence, we will use this value when comparing his experimental data with the present theory.

Normalized Equations

The dimensional equations are normalized by the following procedure. Let

$$F = \frac{g q_o H_o}{\pi}$$

$$Y = H_o + H - x$$

$$V = u_m \cdot b \qquad (24)$$

$$W = u_m \cdot b^2$$

V^2 and W are then proportional to the momentum flux and mass flux, respectively. Eqs. (16) and (17) may now be written

$$\frac{dW}{dY} = -2\alpha V \qquad (25)$$

$$\frac{dV^2}{dY} = -2\frac{F}{Y\left(\frac{V^2}{W(1+\lambda^2)} + u_b\right)} \qquad (26)$$

If V, W and Y have the normalized forms v, w and y we may write

$$V = F^{a_1} \alpha^{a_2} (1+\lambda^2)^{a_3} u_b^{a_4} v$$

$$W = F^{b_1} \alpha^{b_2} (1+\lambda^2)^{b_3} u_b^{b_4} w \qquad (27)$$

$$Y = F^{c_1} \alpha^{c_2} (1+\lambda^2)^{c_3} u_b^{c_4} y$$

and by use of Eqs. (25) and (26) we can identify the powers of Eqs. (27), Hence,

---

* In these experiments the water depth ranged from 2 to 4.5 m and the diameter of the orifice from $5 \cdot 10^{-4}$ to $5 \cdot 10^{-3}$ m.

Eqs. (34) and (35) then have to be integrated from $z = 1$ to $z^{min} = P = \frac{H_o}{H_o + H}$ for a given value of G. P is a scaling parameter and the flow phenomenon is a two parameter problem as indicated by the dimensional analysis. u_m, b and x follow from

$$u_m = (1+\lambda^2) u_b \frac{v^2}{w}$$

$$b = F^{\frac{1}{2}} (1+\lambda^2)^{-1} u_b^{-3/2} \frac{w}{v} \qquad (36)$$

$$x = (H_o + H)(1-z)$$

Eqs. (34) and (35) may be expanded to give

$$\frac{dw}{dz} = -2Gv \qquad (37)$$

$$\frac{dv}{dz} = -\frac{w}{z(v^3 + wv)} \qquad (38)$$

Since $v = w = 0$ at the start of the calculation, that is for $z=1$, we have to find a starting solution for the integration procedure. Similar to the dimensional case we use the simple plume equations. Substituting Eqs. (36) into Eqs. (22) and (23) gives

$$v = \left(\frac{9}{5}\right)^{\frac{1}{3}} G^{\frac{1}{3}} (1-z)^{\frac{2}{3}} \qquad (39)$$

$$w = \frac{6}{5} \left(\frac{9}{5}\right)^{\frac{1}{3}} G^{4/3} (1-z)^{5/3} \qquad (40)$$

These are the starting solutions for the normalized equations. Similar to the dimensional case we have as starting condition that $v^2 >> w$ as for $u_m >> u_b$ which is a restriction to the choice of step length for the numerical integration.

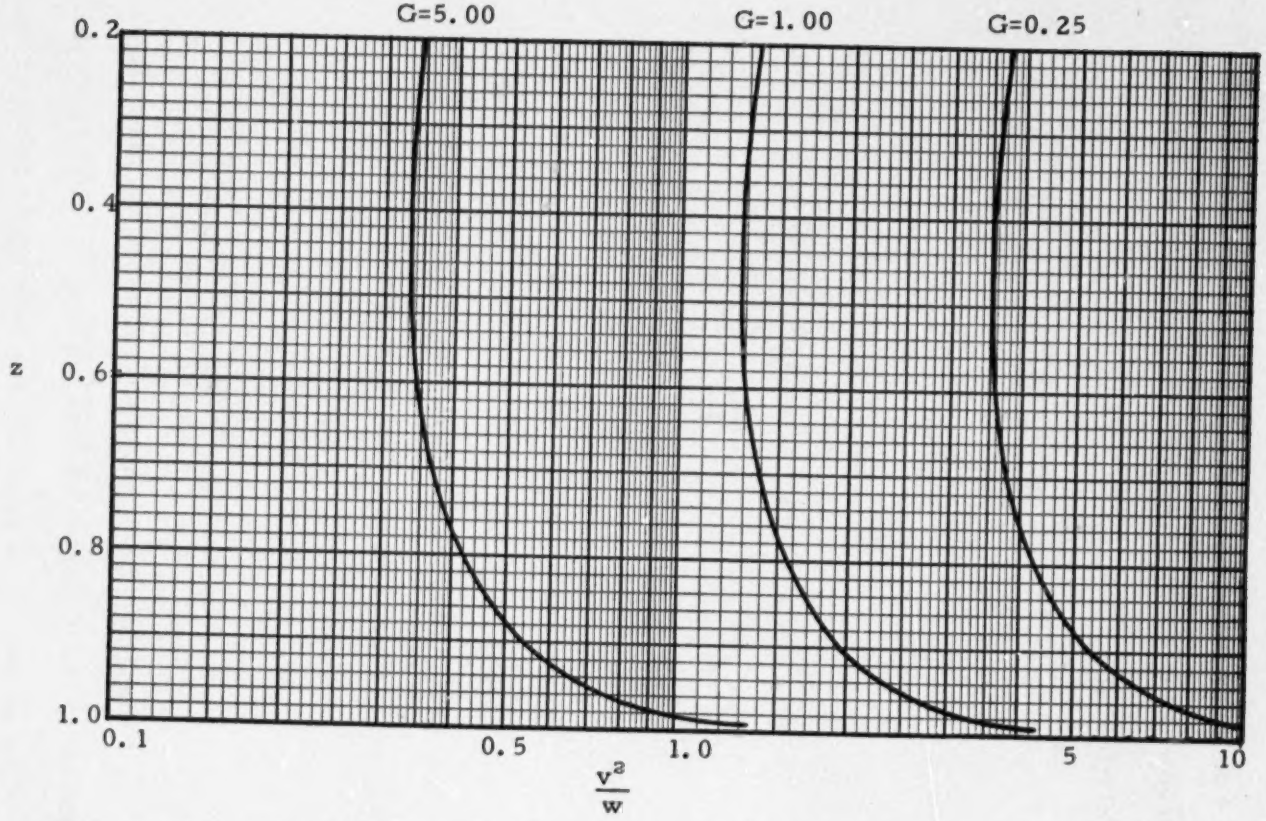


Fig. 2. General solution of the three-dimensional air-bubble plume for G equal 0.25, 1.0 and 5.00. The centerline velocity $u_m = (1+\lambda^2) u_b \frac{v^2}{w}$.

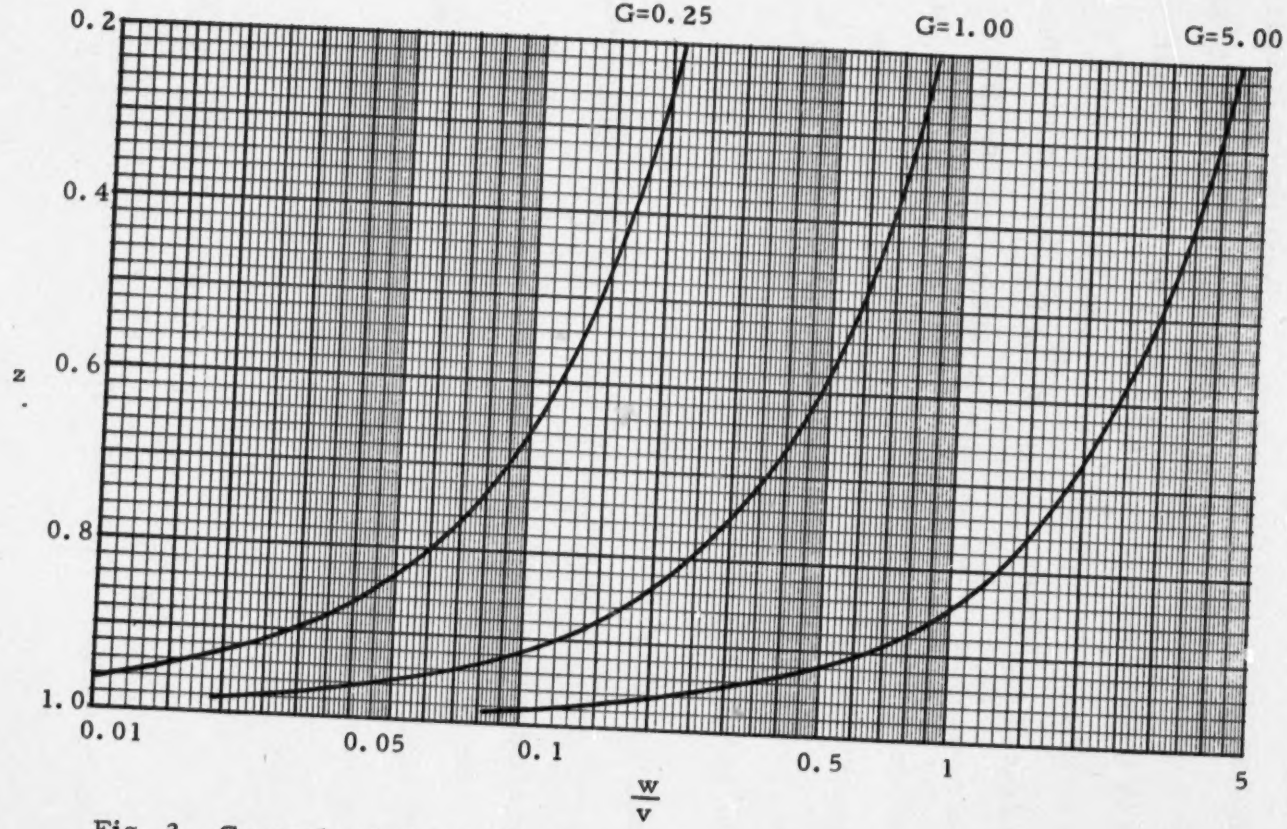


Fig. 3. General solution of the three-dimensional air-bubble plume for G equal 0.25, 1.00 and 5.00. The nominal half-width $b = \left[\frac{g\, q_o\, H_o}{\pi}\right]^{\frac{1}{5}} (1+\lambda^2)^{-1}\, u_b^{-3/2}\, \frac{w}{v}$.

Numerical solutions to Eqs. (37) and (38) with Eqs. (39) and (40) to start the calculation were obtained using an IBM 360/75 digital computer with a subroutine "DEQ/ DIFFERENTIAL EQUATION SOLVER" in FORTRAN IV (LEVEL G) language at the Booth Computing Center of California Institute of Technology. Figs. 2 and 3 show $\frac{v^2}{w}$ and $\frac{w}{v}$ plotted for values of G of 0.25, 1.0 and 5.0 and P equal to 0.20. This is believed to cover most situations of practical interest. Small G values correspond to large flow rates of air, q_o, and small depths, H.

It should be pointed out for the practical use of these graphs that the real source has to be virtually displaced a certain distance Δx which may be assumed to be about 0.8m.

1B. A Line Source in Homogeneous Ambient Fluid

The two-dimensional air-bubble plume – the air-bubble curtain – may be generated by a row of closely spaced orifices. As to the analytical treatment, the basic assumptions and formulation of the problem are similar to the three-dimensional case and the governing equations consequently developed in the same manner.

$$u = u_m \cdot e^{-y^2/b^2} \tag{41}$$

$$(\rho_a - \rho_m) = \Delta\rho_m \cdot e^{-y^2/(\lambda b)^2} \tag{42}$$

are the velocity and density profiles. The volume flux is written

$$Q = \int_{-\infty}^{\infty} u\,dy = \sqrt{\pi}\, u_m b \tag{43}$$

and continuity yields

$$\frac{d}{dx}(u_m b) = \frac{1}{\sqrt{\pi}} 2\alpha u_m \tag{44}$$

The buoyancy flux is given by

$$B = \int_{-\infty}^{\infty} (u+u_b)(\rho_a-\rho_m)\,dy = \sqrt{\pi}\, u_m \Delta\rho_m \frac{\lambda b}{\sqrt{1+\lambda^2}} + \sqrt{\pi}\, u_b \Delta\rho_m \lambda b \tag{45}$$

But

$$B = q_o(\rho_a-\rho_{air})\frac{H_o}{H_o+H-x} \approx \frac{q_o \rho_a H_o}{H_o+H-x} \tag{46}$$

where q_o=air discharge per unit length. This leads to the following buoyancy flux relation

$$\sqrt{\pi}\, u_m \Delta\rho_m \frac{\lambda b}{\sqrt{1+\lambda^2}} + \sqrt{\pi}\, u_b \Delta\rho_m \lambda b = \frac{q_o \rho_a H_o}{H_o+H-x} \tag{47}$$

Similarly, for the momentum flux an equation is written assuming $\rho_m \approx \rho_a$ (Boussinesq assumption)

$$\frac{d}{dx}\left(\frac{u_m^2 b}{\sqrt{2}}\right) = \frac{\Delta\rho_m}{\rho_a} g \lambda b \tag{48}$$

Eliminating $\frac{\Delta\rho_m}{\rho_a}$ from Eqs. (47) and (48) leads to the equations

$$\frac{d}{dx}(u_m b) = \frac{2}{\sqrt{\pi}} \alpha u_m \tag{49}$$

$$\frac{d}{dx}(u_m^2 b) = \frac{\sqrt{2}\, g\, q_o H_o}{\sqrt{\pi}(H_o+H-x)\left[\frac{u_m}{\sqrt{1+\lambda^2}} + u_b\right]} \tag{50}$$

Eqs. (49) and (50) converge to the simple plume relation as $x \to 0$ as $u_m >> u_b$ close to the source. Hence, the following simple plume relations will provide for the required starting conditions

$$\frac{d}{dx}(u_m b) = \frac{2}{\sqrt{\pi}} \alpha u_m \tag{51}$$

$$\frac{d}{dx}(u_m^2 b) = \frac{\sqrt{2}\, q_o H_o \sqrt{1+\lambda^2}}{\sqrt{\pi}\,(H_o+H)\, u_m} \tag{52}$$

Solving Eqs. (51) and (52) we arrive at

$$u_m = \left[\frac{g\, q_o H_o \sqrt{1+\lambda^2}}{\sqrt{2}\,\alpha\,(H_o+H)}\right]^{\frac{1}{3}} \tag{53}$$

$$b = \frac{2}{\sqrt{\pi}} \alpha x \tag{54}$$

These equations are used to give the starting conditions for the numerical integration of Eqs. (49) and (50) reduced for quadrature

$$\frac{du_m}{dx} = -\frac{2\,\alpha\, u_m}{\sqrt{\pi}\, b} + \frac{\sqrt{2}\, g\, q_o H_o}{\sqrt{\pi}\left[\frac{u}{(1+\lambda^2)^{\frac{1}{2}}} + u_b\right](H_o+H-x)\, u_m\, b} \tag{55}$$

$$\frac{db}{dx} = \frac{4\alpha}{\sqrt{\pi}} - \frac{\sqrt{2}}{\sqrt{\pi}} \frac{g\, q_o H_o}{\left[\frac{u_m}{(1+\lambda^2)^{\frac{1}{2}}} + u_b\right](H_o+H-x)\, u_m^2} \tag{56}$$

Normalized Equations

Following the same procedure as for the three-dimensional case we may write the differential equations in normalized form. With

$$V = \left(\frac{\sqrt{2}\, g\, q_o\, H_o}{\sqrt{\pi}}\right) \cdot u_b^{-1} \cdot v = u_m^2 \cdot b$$

$$W = \left(\frac{\sqrt{2}\, g\, q_o H_o}{\sqrt{\pi}}\right) \cdot (1+\lambda^2)^{-\frac{1}{2}} \cdot u_b^{-2}\, w = u_m \cdot b \qquad (57)$$

$$Z = \frac{H_o + H - x}{H_o + H}$$

these equations are

$$\frac{dw}{dz} = -G\frac{v}{w} \qquad (58)$$

$$(v+w)\frac{dv}{dz} = -\frac{w}{z} \qquad (59)$$

where as before v and w represent the dimensionless momentum and mass fluxes and z is a dimensionless length. The source parameter G is defined by

$$G = \sqrt{2}\, (H_o+H)\, (g\, q_o H_o)^{-1}\, \alpha(1+\lambda^2)\, u_b^3 \qquad (60)$$

Eqs. (58) and (59) have to be integrated from z = 1 to z = $P = \frac{H_o}{H_o+H}$ for a given value of G

u_m, b and x are determined by

$$u_m = (1+\lambda^2)^{\frac{1}{2}} u_b \frac{v}{w}$$

$$b = \frac{\sqrt{2}\, g\, q_o\, H_o}{\sqrt{\pi}\,(1+\lambda^2) u_b^3} \frac{w^2}{v} \qquad (61)$$

$$x = (H_o+H)\,(1-z)$$

Similar to the three-dimensional case we assume that the air-bubble equations converge to give the simple plume relations close to the source. This provides us with the following starting conditions for the integration of the normalized equations

$$w = 2^{1/6} G^{2/3}\,(1-z) \tag{62}$$

$$v = 2^{1/3} G^{1/3}\,(1-z) \tag{63}$$

Figs. 4 and 5 show $\frac{v}{w}$ and $\frac{w^2}{v}$ plotted for G values of 0.10, 0.50 and 1.50 and P equal to 0.20. As in the three-dimensional case we have, for the practical use of these graphs, to account for the effect of the virtual displacement of the source.

2A. A Point Source with Zero Slip Velocity.

We can write the conservation equations for the three-dimensional air-bubble plume with zero slip velocity and then combine the momentum flux and buoyancy flux relations. This leads to the following governing equations.

$$\frac{d}{dx}(u_m b^2) = 2\alpha u_m b \tag{64}$$

$$\frac{d}{dx}(u_m^2 b^2) = \frac{2 g q_o H_o (1+\lambda^2)}{\pi (H_o+H-x)\, u_m} \tag{65}$$

With

$$V = (2\alpha)^{\frac{1}{3}}\left[\frac{2}{\sqrt{\pi}}(1+\lambda^2)\right]^{\frac{1}{3}}(g q_o H_o)^{\frac{1}{3}}(H_o+H)^{\frac{1}{3}} v = u_m \cdot b$$

$$W = (2\alpha)^{4/3}\left[\frac{2}{\sqrt{\pi}}(1+\lambda^2)\right]^{\frac{1}{3}}(g q_o H_o)^{\frac{1}{3}}(H_o+H)^{4/3} w = u_m b^2 \tag{66}$$

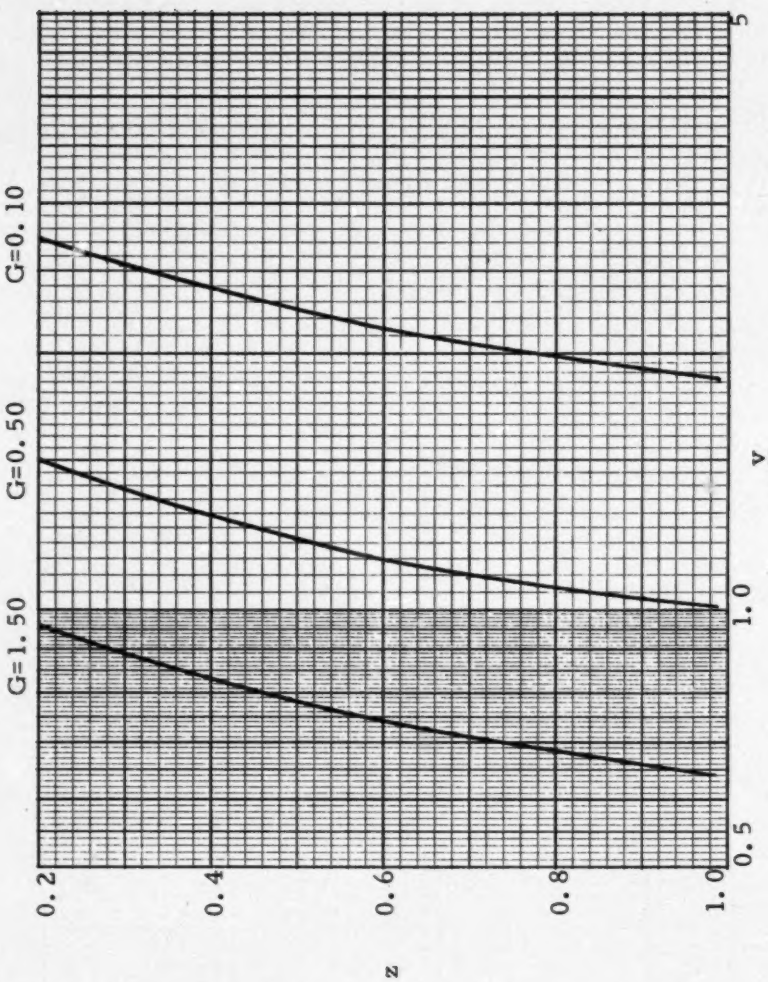


Fig. 4. General solution of the two-dimensional air-bubble plume for G equal 0.10, 0.50 and 1.50. The centerline velocity

$$u_m = (1+\lambda^3)^{\frac{1}{2}} u_b \frac{v}{w} .$$

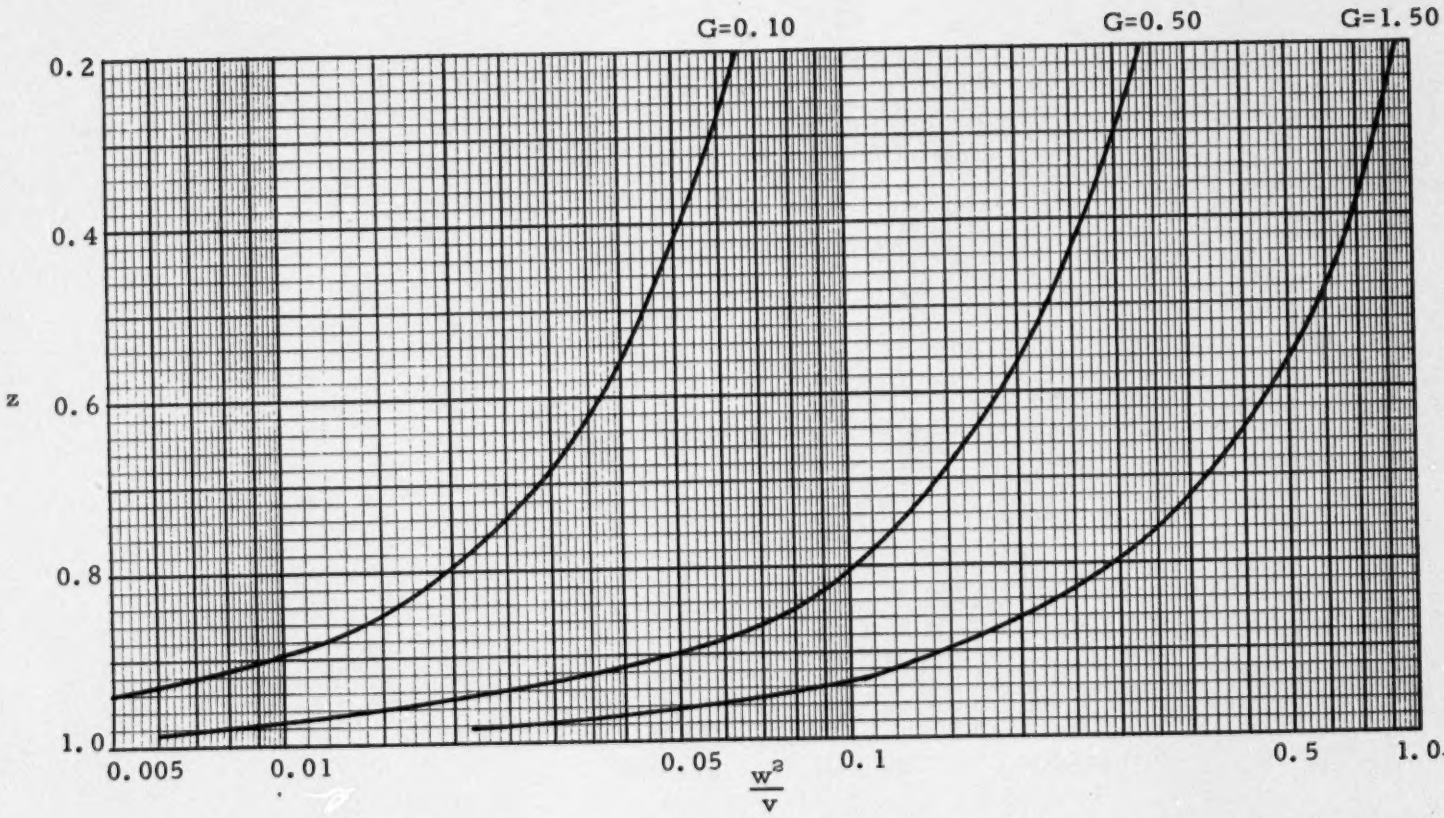


Fig. 5. General solution of the two-dimensional air-bubble plume for G equal 0.1, 0.5 and 1.5.

The nominal half-width $b = \frac{\sqrt{2}\, g\, q_o\, H_o}{\sqrt{\pi}\,(1+\lambda^2)\, u_b^3}\, \frac{w^2}{v}$.

$$z = \frac{H_o + H - x}{H_o + H}$$

the normalized equations take the form

$$\frac{dw}{dz} = -v \tag{67}$$

$$\frac{dv^2}{dz} = -\frac{w}{zv^2} \tag{68}$$

u_m, b and x are determined by

$$u_m = \frac{(g\, q_o\, H_o)^{\frac{1}{3}} (1+\lambda^2)^{\frac{1}{3}}}{(2\pi)^{\frac{1}{3}}\, \alpha^{\frac{2}{3}}\, (H_o+H)^{\frac{2}{3}}} \frac{v^2}{w}$$

$$b = 2\alpha\,(H_o+H)\,\frac{w}{v} \tag{69}$$

$$x = (H_o+H)\,(1-z)$$

The normalized equations are solved numerically with the starting conditions derived from the simple plume relations in normalized form

$$w = \frac{3}{5}\left(\frac{27}{60}\right)^{\frac{1}{3}} (1-z)^{5/3} \tag{70}$$

$$v = \left(\frac{27}{60}\right)^{\frac{1}{3}} (1-z)^{\frac{2}{3}} \tag{71}$$

Figs. 6 and 7 show the normalized solution of the three-dimensional air-bubble plume with zero slip velocity.

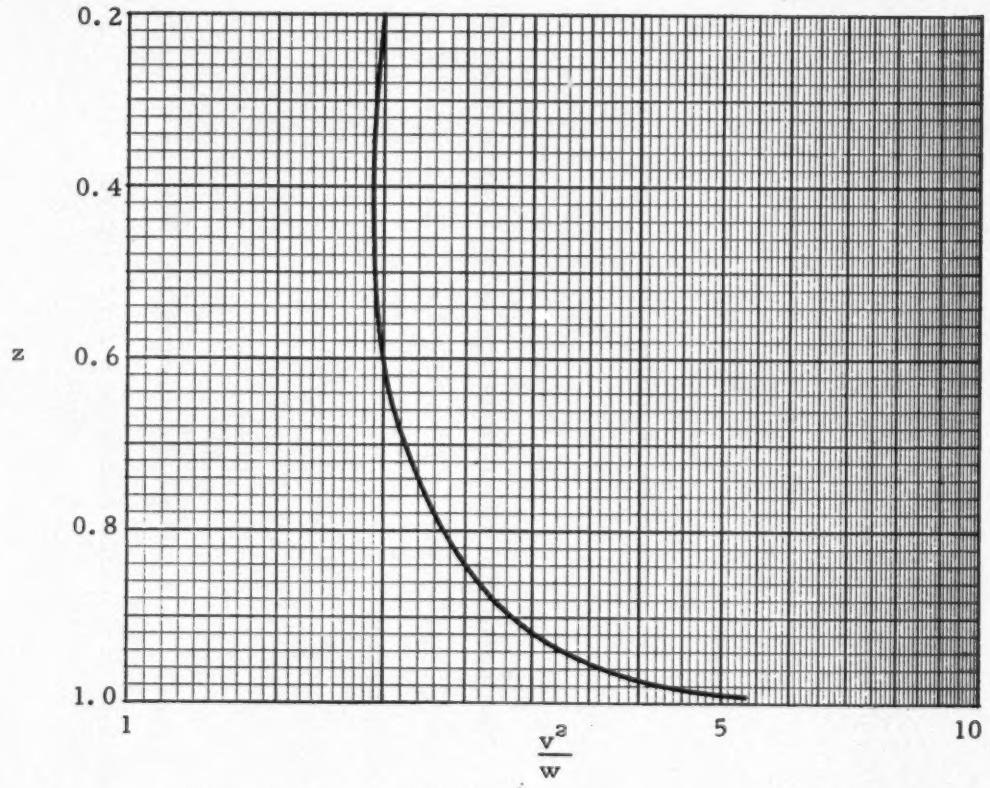


Fig. 6. General solution for the three-dimensional air-bubble plume with zero slip velocity. The centerline velocity

$$u_m = \frac{(g\, q_o\, H_o)^{\frac{1}{3}} (1+\lambda^2)^{\frac{1}{3}}}{(2\pi)^{\frac{1}{3}}\, \alpha^{\frac{2}{3}}\, (H_o+H)^{\frac{5}{6}}} \frac{v^2}{w} .$$

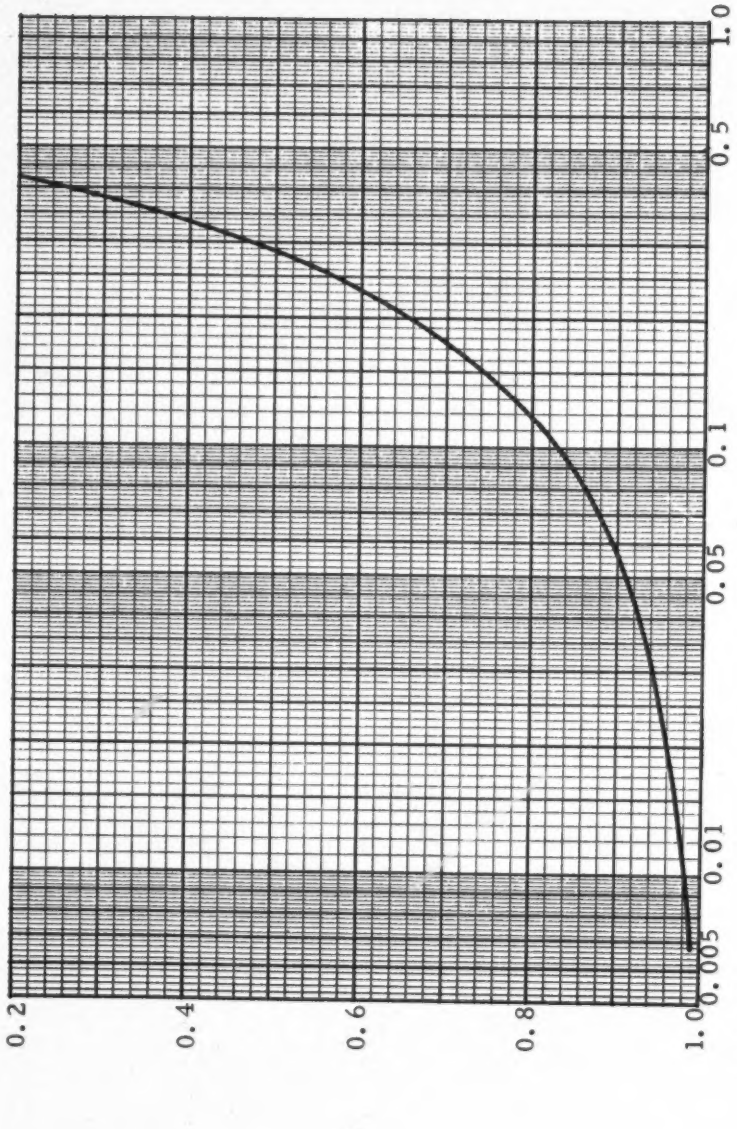


Fig. 7. General solution of the three-dimensional air-bubble plume with zero slip velocity. The nominal half-width $b = 2\alpha(H_o+H)\frac{w}{v}$.

2B. A Line Source with Zero Slip Velocity.

The governing equations are

$$\frac{d}{dx}(u_m b) = \frac{2}{\sqrt{\pi}} \alpha u_m \tag{72}$$

$$\frac{d}{dx}(u_m^2 b) = \frac{\sqrt{2} g q_o H_o (1+\lambda^2)^{\frac{1}{2}}}{\sqrt{\pi}(H_o+H-x) u_m} \tag{73}$$

With the following definitions

$$V = \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\frac{1}{3}} (1+\lambda^2)^{\frac{1}{3}} \left(\frac{\sqrt{2}}{\sqrt{\pi}} g q_o H_o\right)^{\frac{2}{3}} (H_o+H)^{\frac{1}{3}} v = u_m^2 b$$

$$W = \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\frac{2}{3}} (1+\lambda^2)^{1/6} \left(\frac{\sqrt{2}}{\sqrt{\pi}} g q_o H_o\right)^{\frac{1}{3}} (H_o+H)^{\frac{2}{3}} w = u_m \cdot b \tag{74}$$

$$z = \frac{H_o+H-x}{H_o+H}$$

the normalized equations may be written

$$\frac{dw}{dz} = -\frac{v}{w} \tag{75}$$

$$\frac{dv}{dz} = -\frac{w}{zv} \tag{76}$$

u_m, b and x are given by

$$u_m = \frac{(g q_o H_o)^{\frac{1}{3}} (1+\lambda^2)^{1/6}}{(\sqrt{2}\alpha)^{\frac{1}{3}} (H_o+H)^{\frac{1}{3}}} \frac{v}{w}$$

$$b = \frac{2}{\sqrt{\pi}} \alpha (H_o+H) \frac{w^2}{v} \tag{77}$$

$$x = (H_o+H)(1-z)$$

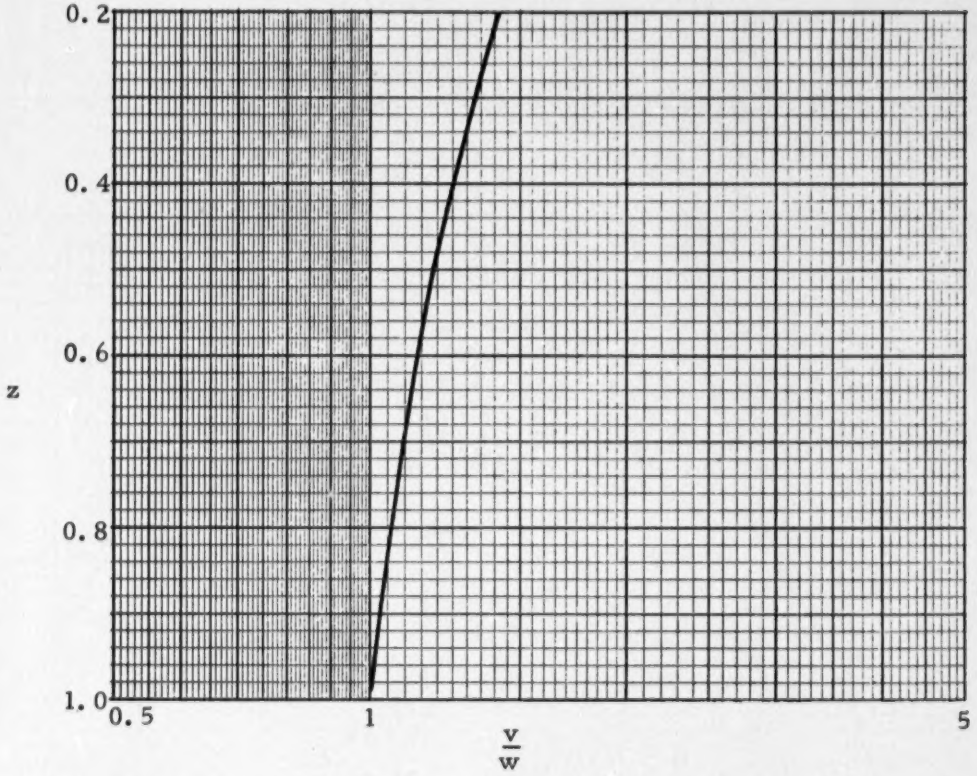


Fig. 8. General solution of the two-dimensional air-bubble plume with zero slip velocity. The centerline velocity

$$u_m = \frac{(g\, q_o\, H_o)^{\frac{1}{3}}(1+\lambda^2)^{1/6}}{(\sqrt{2}\alpha)^{\frac{1}{3}}(H_o+H)^{\frac{1}{3}}}\,\frac{v}{w}\,.$$

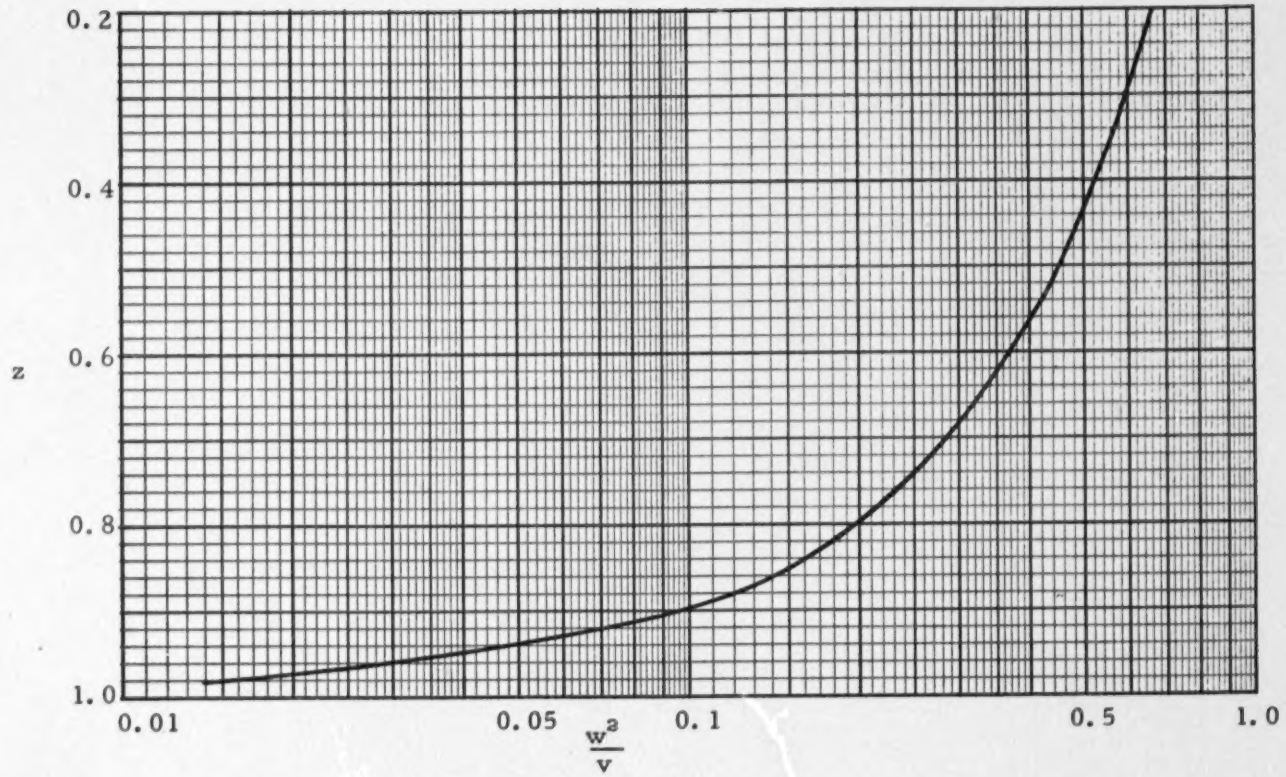


Fig. 9. General solution of the two-dimensional air-bubble plume with zero slip velocity. The nominal half-width

$$b = \frac{2}{\sqrt{\pi}} \alpha(H_o+H) \frac{w^2}{v} .$$

There is no analytical solution to Eqs. (75) and (76) in closed form which takes the boundary conditions v = w = 0 for z = 1. Starting conditions for the numerical integration are provided for by the simple plume relations in normalized form

$$w = 2^{1/6}(1-z) \tag{78}$$

$$v = 2^{1/3}(1-z) \tag{79}$$

Eqs. (75) and (76) may now be integrated from z=1 to $z=P=\frac{H_o}{H_o+H}$. Figs.8 and 9 show the normalized solution of the two-dimensional air-bubble plume with zero slip velocity.

NEGLECTING THE EFFECT OF COMPRESSIBILITY

For the sake of completeness, let us also discuss a situation where the lighter fluid released from the source has no compressibility. If the two fluids are immiscible, bubbles will be produced at the source just as in the air-bubble case. The governing equations for a point source of buoyancy only are then with the source strength $q_o H_o/(H_o+H)$

$$\frac{d}{dx}(u_m b^2) = 2\alpha u_m b \tag{80}$$

$$\frac{d}{dx}(u_m^2 b^2) = \frac{2g q_o H_o}{\pi (H_o+H)\left(\frac{u_m}{1+\lambda^2} + u_b\right)} \tag{81}$$

Following the same normalization procedure as before we have

$$V = (2\alpha)^{-1}(1+\lambda^2)^{-1}u_b^{-2}\left[\frac{2g q_o H_o}{\pi(H_o+H)}\right] v = u_m b$$

$$W = (2\alpha)^{-2}(1+\lambda^2)^{-3}u_b^{-5}\left[\frac{2g q_o H_o}{\pi(H_o+H)}\right]^2 w = u_m b^2 \tag{82}$$

$$x = (2\alpha)^{-2}(1+\lambda^2)^{-2}u_b^{-3}\left[\frac{2 g q_o H_o}{\pi(H_o+H)}\right] z$$

and the differential equations

$$\frac{dw}{dz} = v \tag{83}$$

$$(v^2+w)\frac{dv^2}{dz} = w \tag{84}$$

As there is no analytical solution in closed form to Eqs. (83) and (84) that takes the boundary conditions v=w=0 at z=0, the equations have to be solved numerically. The starting conditions are obtained from the simple plume relations

$$v = \left(\frac{27}{60}\right)^{\frac{1}{3}} z^{\frac{2}{3}} \tag{85}$$

$$w = \frac{3}{5}\left(\frac{27}{60}\right)^{\frac{1}{3}} z^{5/3} \tag{86}$$

The solution is not given here. The two-dimensional case may be treated similarly.

Example

To exemplify the use of the present theory, take the case of a point source in homogeneous water with the following characteristical data

$$q_o = 4 \cdot 10^{-3}\ m^3/s$$

$$H = 10\ m$$

$$H_o = 10.4\ m$$

We want to know the centerline velocity, u_m, the nominal half-width, b, and the generated vertical flow, Q, at say 5 m above the source.

Let us assume the following values of λ, α and u_b

$$\lambda = 0.2$$

$$\alpha = 0.07 \quad \text{(See Fig. 17)}$$

$$u_b = 0.30 \text{ m/s}$$

The source parameter G, defined by Eq. (33), then takes the value G = 0.68 and the normalized height above the mathematical origin, Eq. (32), is $z = \frac{20.4-5}{20.4+0.8} = 0.73$ Figs. 2 and 3 give $\frac{v^2}{w} = 1.80$ and $\frac{w}{v} = 0.25$. u_m and b then follow from Eqs. (36) and $Q = \pi u_m b^2$ can be calculated.

COMPARISON WITH EXPERIMENTAL DATA

Table 1 sums up the air-bubble plume cases that have been analyzed. To verify the theory and to determine the proper choice of values of the constants included in the theory – primarily the entrainment coefficient α and the spreading ratio λ – the analytical results are compared with experimental data. Many experimental studies of air-bubble systems reported in the literature are not very comprehensive or consistent. A laboratory study of large scale air-bubble plumes by Kobus (1968) covers both the two- and three-dimensional cases. The present theory is compared with the results given by Kobus.

Typical measurements of the centerline velocity and the lateral velocity distribution from two different experiments with a point source are plotted together with the theory in Figs. 10 to 12. For the theoretical calculations α values were chosen to give the best fit between observed and predicted rate of growth of the nominal plume half-width. The lateral spreading of the air-bubbles is slow relative to the expansion of the plume. Hence, the spreading ratio λ is considerably less than one. Laboratory observations* indicate that λ is fairly constant throughout the rise of the plume, the order of magnitude about 0.2. Since theoretical calculations based on λ=0.1, 0.2 or 0.3 give practically the same result we have used a constant value 0.2 in this study. For reasons that we will discuss later, the slip velocity, u_b, was also held constant and equal to 0.30 m/s when comparing theoretical and experimental results for the three-dimensional air-bubble plumes. The experimental data

* Photographical recordings of air-bubble plumes. The experiments were conducted at Chalmers Institute of Technology.

Table 1. Air-Bubble Plumes in Homogeneous Environment

Case	Normalized Differential Eqs.	Starting Conditions	Parameters and Boundary Conditions
1A. General Case, Point Source	$\frac{dw}{dz} = -2Gv$ $\frac{dv}{dz} = \frac{w}{z(v^3 + wv)}$	$w=\frac{6}{5}\left(\frac{9}{5}\right)^{\frac{1}{3}} G^{4/3}(1-z)^{5/3}$ $v = \left(\frac{9}{5}\right)^{\frac{1}{3}} G^{1/3}(1-z)^{2/3}$ $v^2 > w$	$G=\pi^{\frac{1}{2}} \frac{(H_o+H)\,\alpha\,(1+\lambda^2)\,u_b^{3/2}}{(g\,q_o\,H_o)^{\frac{1}{2}}}$ $z = 1$ to $z = P = H_o/(H_o+H)$
1B. General Case, Line Source	$\frac{dw}{dz} = -G\frac{v}{w}$ $\frac{dv}{dz} = -\frac{w}{z(v+w)}$	$w=2^{1/6} G^{2/3}(1-z)$ $v=2^{1/3} G^{1/3}(1-z)$ $v > w$; $G < \sqrt{2}$	$G=2^{\frac{1}{2}} \frac{(H_o+H)\,\alpha\,(1+\lambda^2)\,u_b^3}{g\,q_o\,H_o}$ $z = 1$ to $z = P = H_o/(H_o+H)$
2A. Zero Slip Velocity, Point Source	$\frac{dw}{dz} = -v$ $\frac{dv}{dz} = -\frac{w}{2z\,v^3}$	$w=\frac{3}{5}\left(\frac{27}{60}\right)^{\frac{1}{3}}(1-z)^{5/3}$ $v=\left(\frac{27}{60}\right)^{\frac{1}{3}}(1-z)^{2/3}$	$z=1$ to $z=P = H_o/(H_o+H)$
2B. Zero Slip Velocity, Line Source	$\frac{dw}{dz} = -\frac{v}{w}$ $\frac{dv}{dz} = -\frac{w}{zv}$	$w = 2^{1/6}(1-z)$ $v = 2^{1/3}(1-z)$	$z=1$ to $z=P = H_o/(H_o+H)$

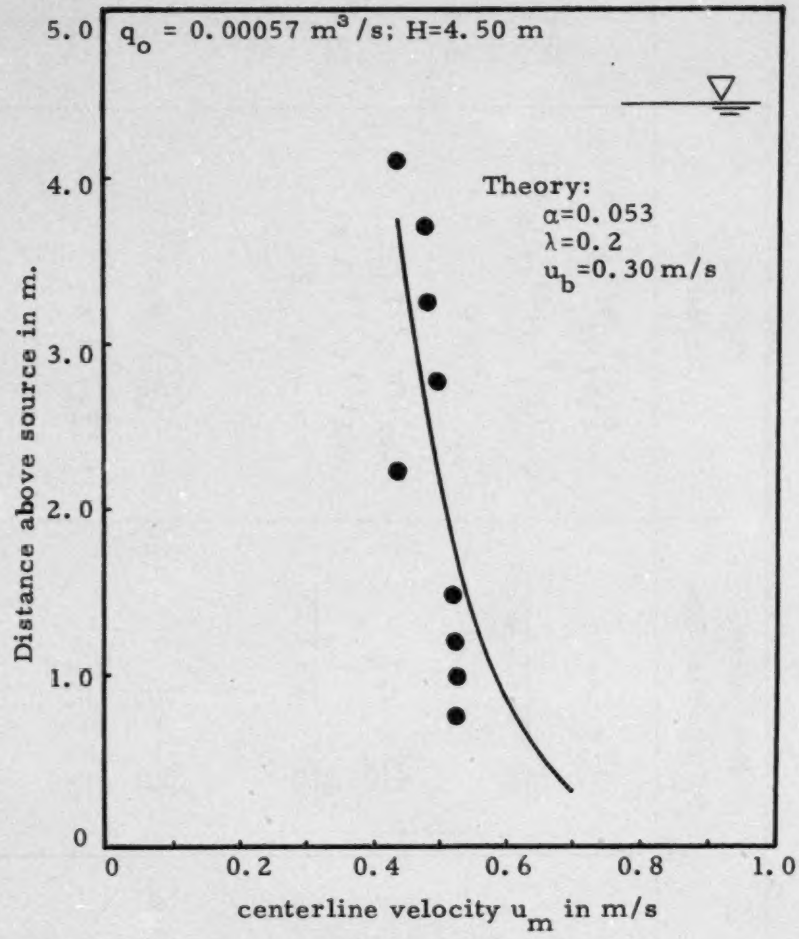


Fig. 10. Observed and predicted variation of centerline velocity with distance above the (real) source. Experimental data given by Kobus (1968).

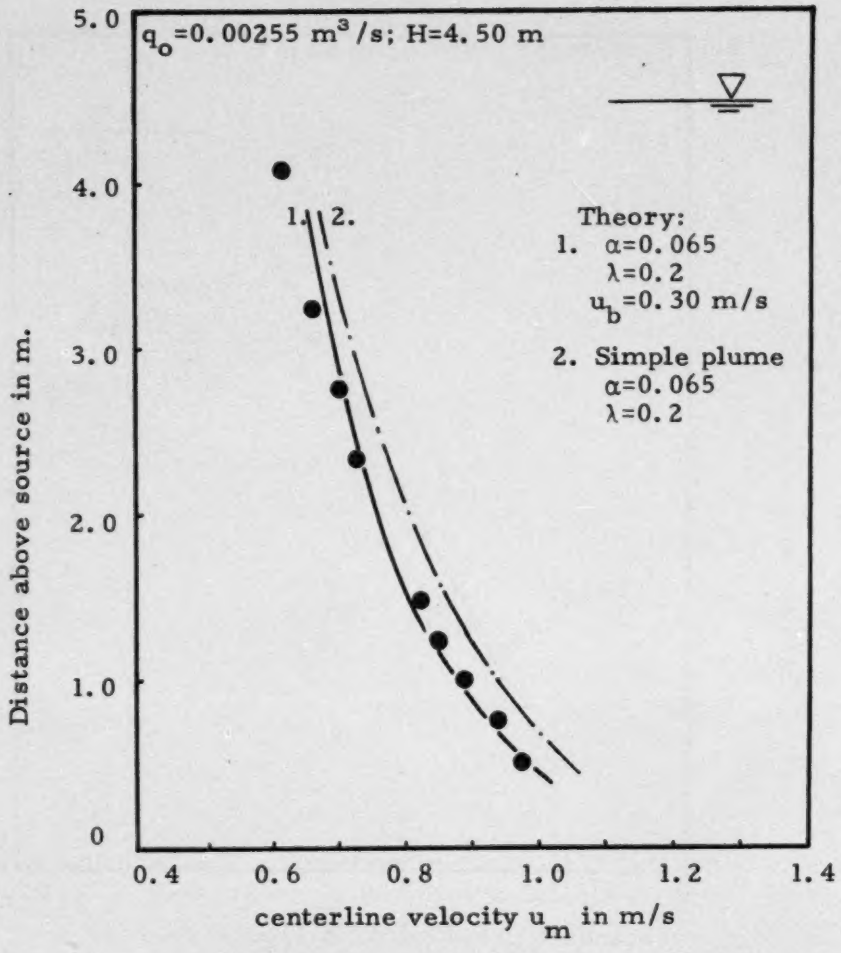


Fig. 11. Observed and predicted variation of centerline velocity with distance above the (real) source. Experimental data given by Kobus (1968).

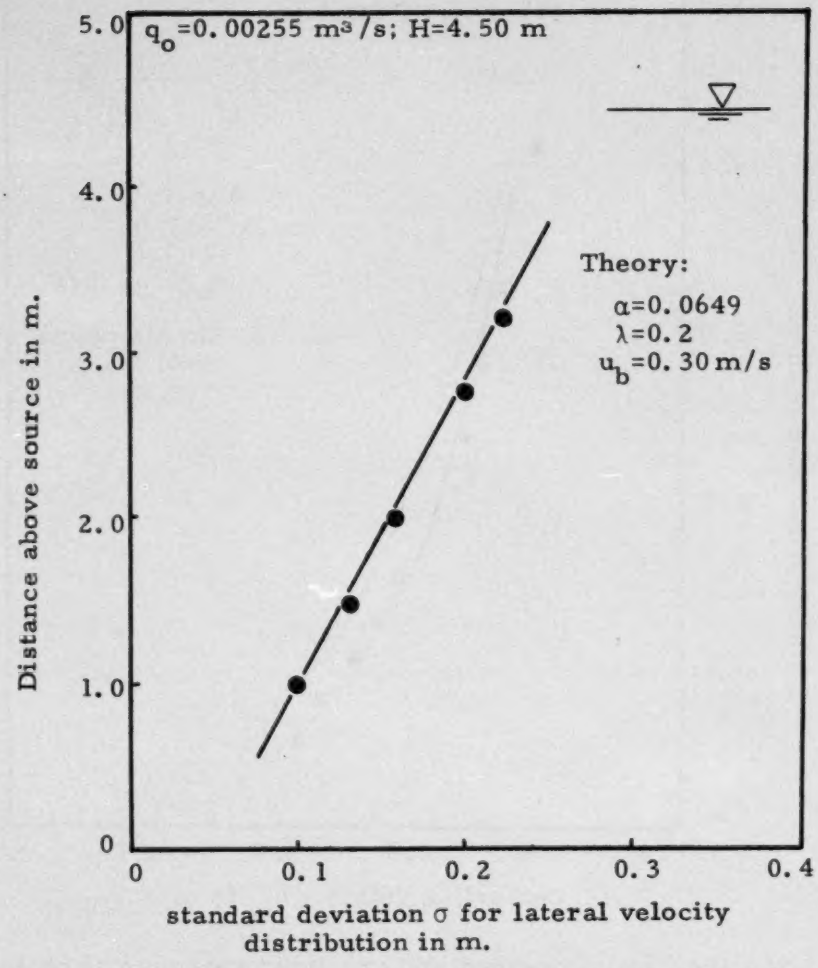


Fig. 12. Observed and predicted rate of growth of the lateral velocity profile with the distance above the (real) source. Experimental data by Kobus (1968).

provide a strong support for the present theoretical approach as demonstrated by the two test runs in Figs. 10 to 12.

Measured centerline velocities of air-bubble plumes from a single point source show little scattering and seem to be consistent. It is unfortunately difficult to get reliable data when conducting similar tests with air-bubble screens. One of the reasons for this is the fact that slow internal seiches tend to develop in the laboratory tank induced by the air-bubble system itself. Kobus has reported integration over periods as long as 5 minutes which of course tends to reduce the measured velocities if the air-bubble plume is oscillating. Figs. 13 to 16 show results from experiments with air-bubble curtains. Observed values of the centerline velocity seem to be somewhat lower than predicted velocities. This, however, can be reduced to a problem of curve fitting since the characteristic properties of the two-dimensional air-bubble plume expressed by the theory is supported by the experimental data just as in the three-dimensional case.

Measured centerline velocities are compared to the theory using three sets of different values of α, λ and u_b. The first calculation has $\lambda=0.2$, $u_b=0.30$ m/s and a value of α that makes the predicted expansion of the plume agree with the observed plume growth. For the second calculation u_b was estimated as the difference between the rise velocity of the air-bubbles – also measured by Kobus – and the observed centerline velocity of the plume water as these two velocities are relatively constant. We have finally increased the α value to 0.16 in

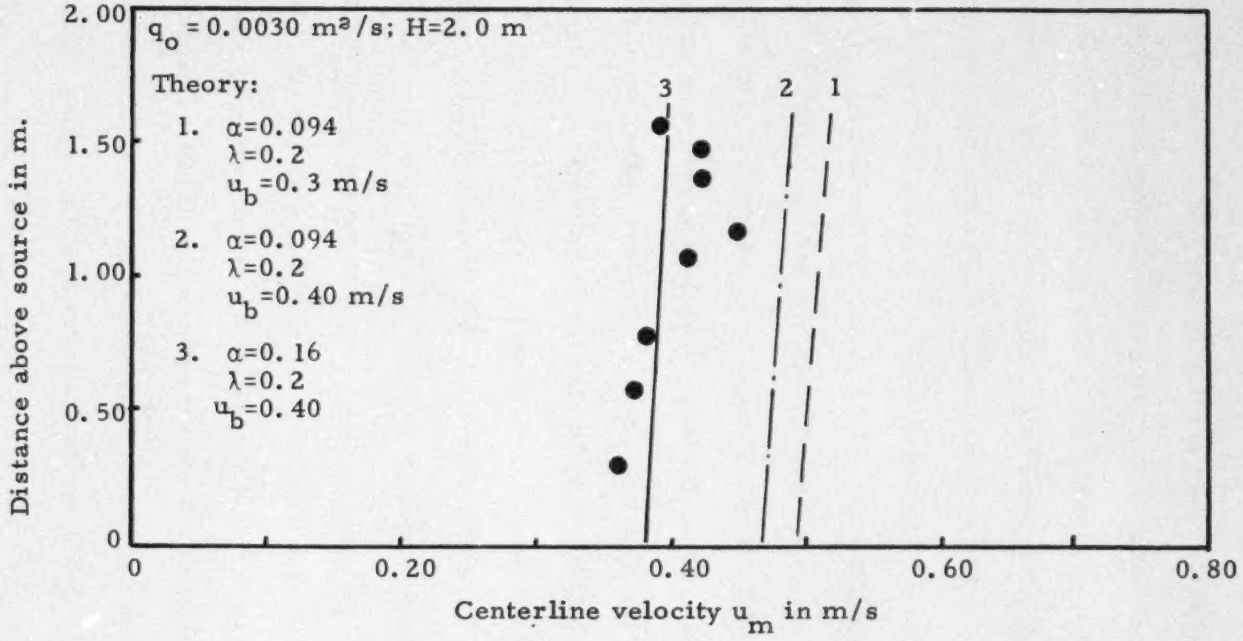


Fig. 13 Observed and predicted variation of centerline velocity with distance above the (real) source. Experimental data given by Kobus (1968).

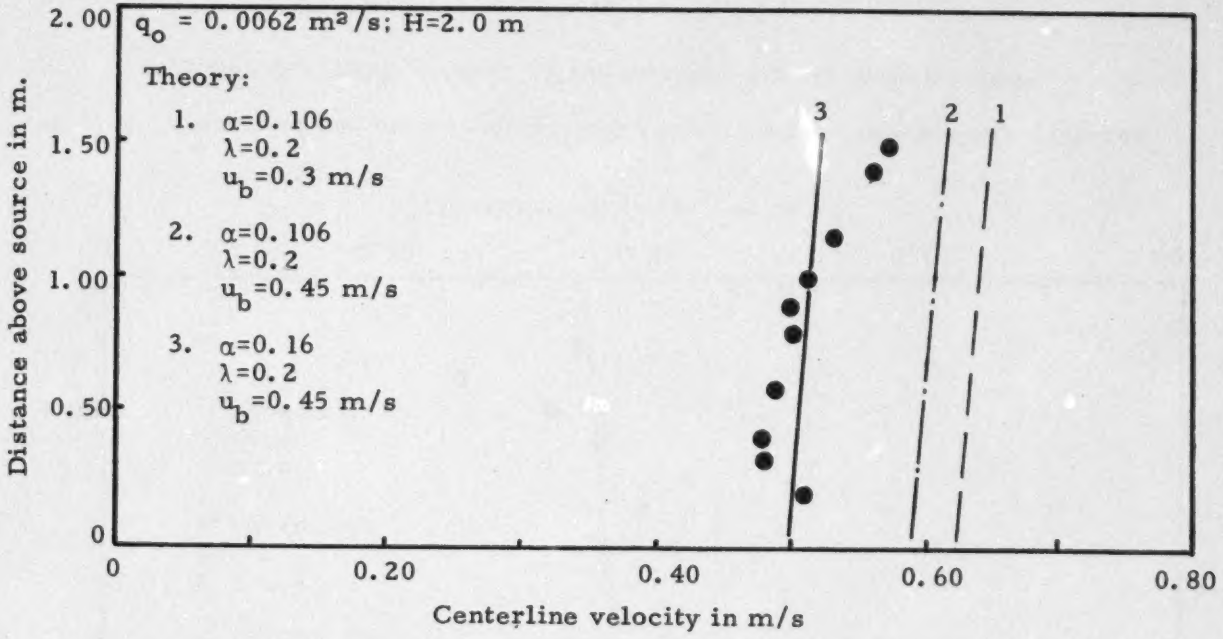


Fig. 14 Observed and predicted variation of centerline velocity with distance above the (real) source. Experimental data given by Kobus (1968).

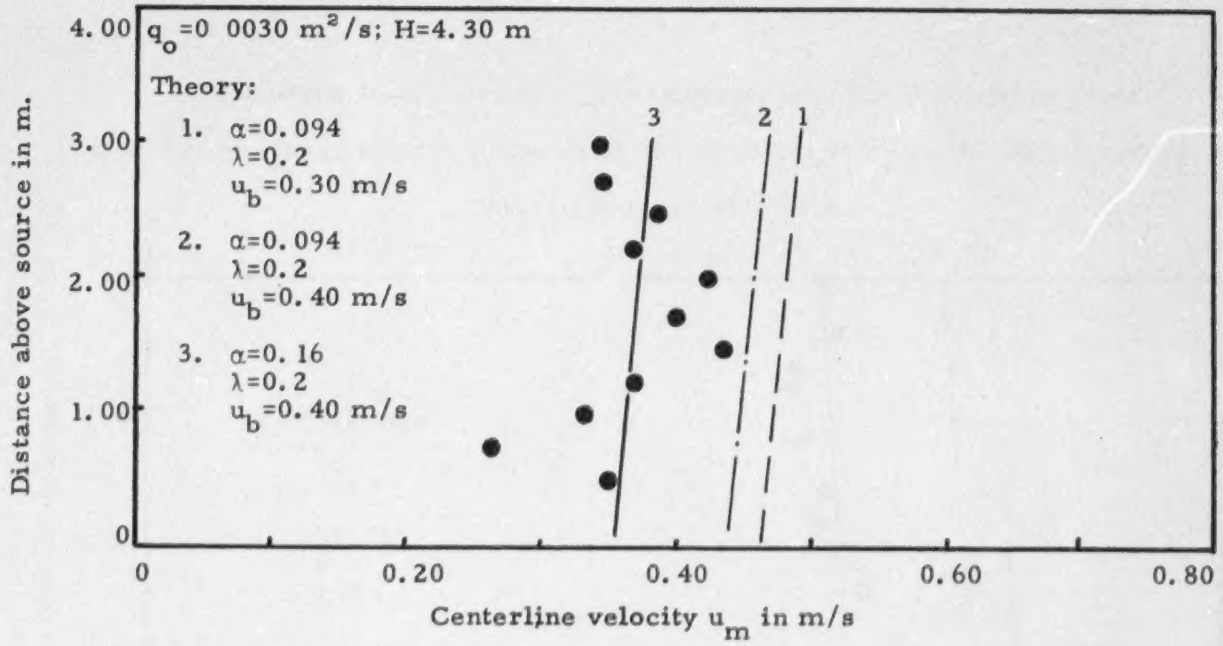


Fig. 15 Observed and predicted variation of centerline velocity with distance above the (real) source. Experimental data by Kobus (1968).

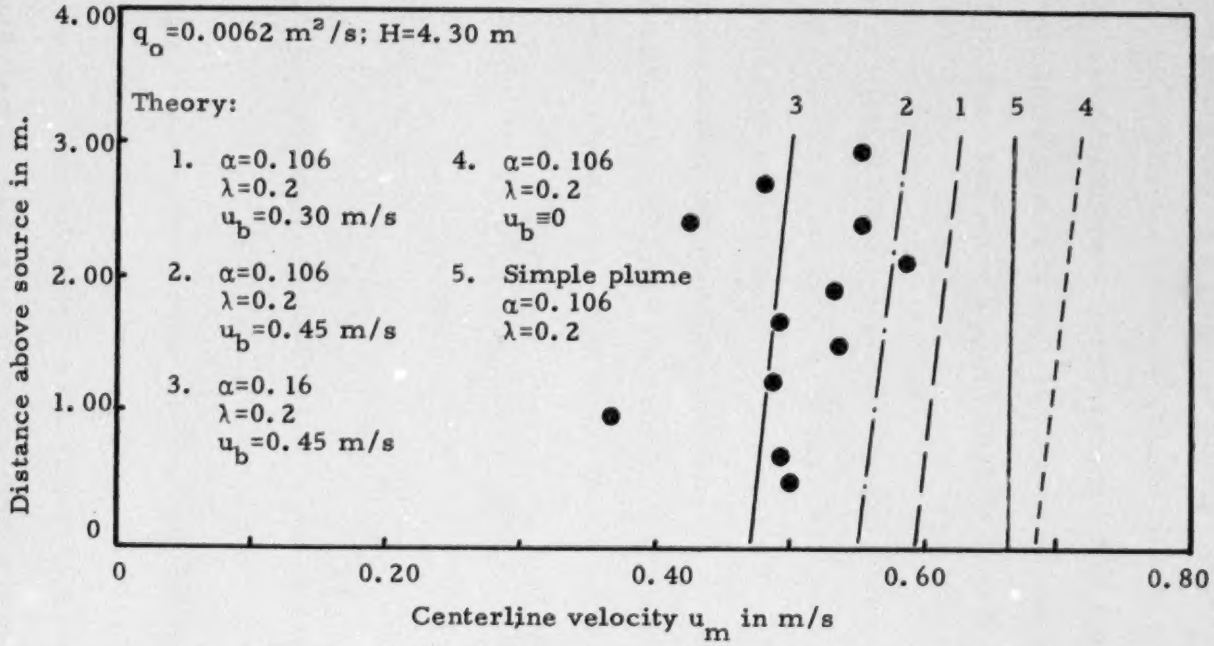


Fig. 16 Observed and predicted variation of centerline velocity with distance above the (real) source. Experimental data by Kobus (1968).

which case the observed and predicted velocities match each other very well but not the variation of the nominal plume half-width. Fig. 16 also shows the centerline velocity assuming a simple plume – no compressibility and no slip velocity – and the velocity in the case of zero slip velocity but with compressibility – small bubbles. Before making any conclusions we would first like to review some empirical background for the proper choice of values of the constants included in the theory.

It has been proved experimentally by Kobus that the rate of plume expansion characterized by $\frac{db}{dx}$ is practically a constant, slightly a function of the rate of air flow, q_o, but independent of the source diameter in the range covered by the experiments, that is from 0.05 mm to 0.5 mm. The theoretical calculations reveal that b is essentially a function of α only. This means that a good estimate of the α value is

$$\alpha = \frac{5}{6}\frac{db}{dx} = \frac{5\sqrt{2}}{6}\frac{d\sigma}{dx} \tag{87}$$

for the three-dimensional case and

$$\alpha = \frac{\sqrt{\pi}}{2}\frac{db}{dx} = \frac{\sqrt{\pi}}{\sqrt{2}}\frac{d\sigma}{dx} \tag{88}$$

for the two-dimensional case. Hence using the data given by Kobus we can plot α as a function of q_o, see Fig. 17.

The α-values for the case of a point source indicate a growth with the air flow rate reaching asymptotically a value of about 0.08. It is of interest to note that Rouse, Yih and Humphreys (1952) found α=0.082 for the case of a three-dimensional simple plume. Furthermore, experimental data by Lee and Emmons (1961) suggest an α-value of

about 0. 16 for a buoyant line source. Although the air curtain experiments reported by Kobus only cover a relatively narrow range in q_o, it is possible that there is an increase of the α-value with q_o, very similar to the three-dimensional case, with α reaching an asymptotic value of about 0. 16 for very large air flow rates.

Studies on the rise velocity of single air-bubbles in still water show that there is a wide range in bubble sizes that have terminal velocities of the order of 0. 3 m/s, see for instance Haberman and Morton (1954). Hence, it is believed that u_b=0. 30 m/s is a proper value to use in most practical situations unless the discharge rate per orifice is exceptionally small as for instance in the case of a porous hose. As demonstrated by line 4 in Fig. 16 the buoyancy flux is more effectively used for generating a plume when the bubbles are small. Hence, the porous hose or a similar device producing very small air-bubbles, represent the ideal design of an air-bubble system.

To summarize we have found that the experimental data support the proposed theory and there is a very good agreement between theory and observations in the three-dimensional case if proper values of the coefficients are used. In the two-dimensional case we have to choose large values of either the slip velocity or the entrainment coefficient to match theory and experiments. We are inclined to believe that the reason for this is measuring difficulties in the case of the air-bubble screen resulting in too low values of the observed centerline velocities reported by Kobus.

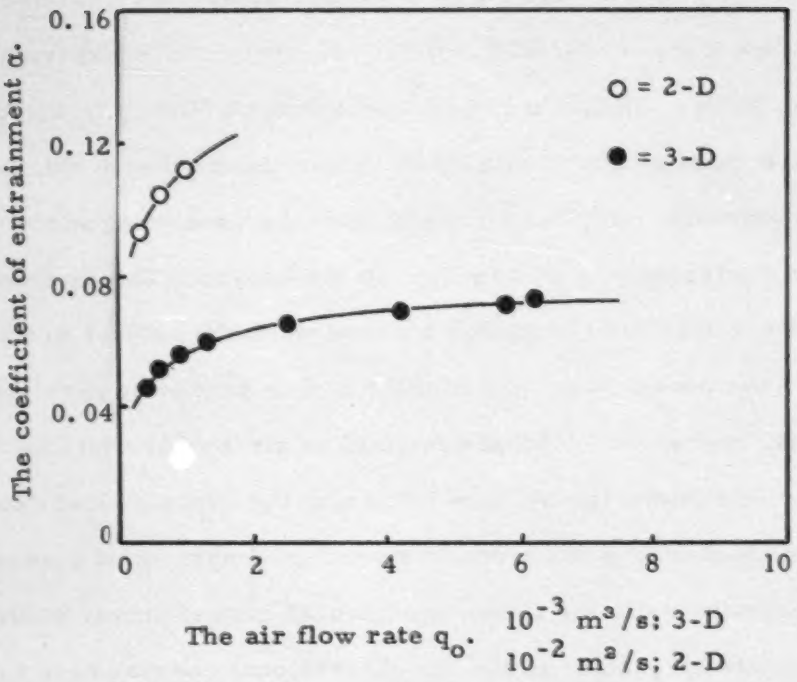


Fig. 17. The coefficient of entrainment as a function of the air flow rate for two- and three-dimensional air-bubble plumes. Experimental data by Kobus (1968).

AIR BUBBLE SYSTEMS IN STRATIFIED WATER

Air-bubble systems are likely to be used extensively in the future for water quality management of lakes and reservoirs. Hence, a theory that predicts the circulation and mixing induced by air-bubble plumes in stratified water is very much needed. Preliminary experiments have been carried out by the senior author at Chalmers Institute of Technology to study this problem and a continuation of this investigation is planned.

The presence of a density stratification in the water gives rise to an uncoupling effect between the water flow and the air-bubbles, as shown schematically in Fig. 18 for a two-layer system. A continuous stratification may produce a similar effect.

Let us confine this discussion to a linear density profile of the ambient water and hypothesize that the governing plume equations may be developed by the same integral procedure as previously used for homogeneous water. This proposed analysis will be limited to the two-dimensional case, that is an air-curtain. The lateral velocity distribution is presumed as before to be Gaussian:

$$u = u_m \cdot e^{-y^2/b^2} \tag{89}$$

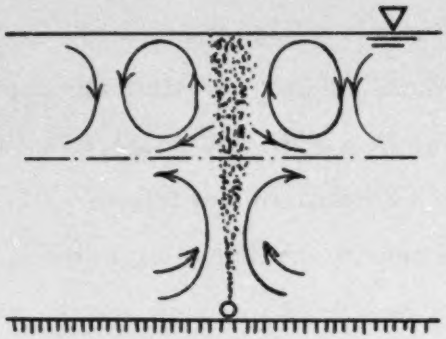
and the local density of water within the plume, ρ_w, is assumed to have a similar variation

$$\rho_w - \rho_a = \Delta\rho_w \cdot e^{-y^2/b^2} \tag{90}$$

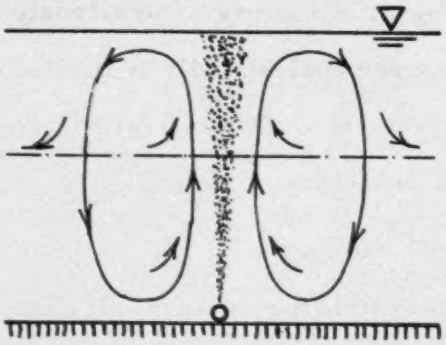
where ρ_a is the ambient density at that particular level.

If c is the local concentration of air by volume we consider the lateral distribution within the plume to be given by

$$c = c_m e^{-y^2/(\lambda b)^2} \tag{91}$$



a. Complete uncoupling between plume water and air-bubbles at the interface.



b. Partial uncoupling at the interface.

Fig. 18. Observed possible flow regimes with a bubble screen in a two-layer density system.

The density of the air-water mixture, ρ_m, is then

$$\rho_m = c\,\rho_{air} + (1-c)\,\rho_w \tag{92}$$

The Boussinesq approximation holds for mass and momentum flux relations: $\rho_m \approx \rho_w \approx \rho_o$ since $c << 1$ and $\frac{\rho_{air}}{\rho_w} = O(10^{-3})$. ρ_o is the reference density chosen as the density of the ambient water at the source level.

The following equations of conservation may then be written for a line source.

1. For the volume (mass) flux

$$\frac{d}{dx}\int_{-\infty}^{\infty} \rho_m\, u\, dy = 2\alpha\, u_m\, \rho_a \tag{93}$$

which is reduced to

$$\frac{d}{dx}(u_m b) = \frac{2}{\sqrt{\pi}}\alpha\, u_m \tag{94}$$

The coefficient of entrainment, α, is as before assumed to be constant.

For the air flow we have a similar relation

$$\int_{-\infty}^{\infty} c(u+u_b)\, dy = \frac{q_o H_o}{H_o + H - x} \tag{95}$$

which may be written

$$c_m\left(\frac{u_m \lambda b}{\sqrt{1+\lambda^2}} + u_b \lambda b\right) = \frac{1}{\sqrt{\pi}}\,\frac{q_o H_o}{H_o + H - x} \tag{96}$$

2. The momentum flux equation is

$$\frac{d}{dx}\int_{-\infty}^{\infty} \rho_m\, u^2\, dy = \int_{-\infty}^{\infty} g\,(\rho_a - \rho_m)\, dy \tag{97}$$

From Eq. (92) follows

$$\rho_a - \rho_m = \rho_a - c\,\rho_{air} - (1-c)\,\rho_w = (\rho_a - \rho_w) + c(\rho_w - \rho_{air}) \qquad (98)$$

which is approximated to

$$\rho_a - \rho_m = (\rho_a - \rho_w) + c\,\rho_o \qquad (99)$$

Hence, Eq. (97) may be written

$$\frac{d}{dx}\int_{-\infty}^{\infty} \rho_o\, u^2\, dy = \int_{-\infty}^{\infty} g\left[(\rho_a - \rho_w) + c\,\rho_o\right] dy \qquad (100)$$

which is reduced to

$$\frac{d}{dx}(u_m^2\, b) = \sqrt{2}\, g\, c_m\, \lambda b - \sqrt{2}\, g\, \frac{\Delta\rho_w}{\rho_o}\, b \qquad (101)$$

3. The conservation of buoyancy flux.

The buoyancy flux relative to the reference density ρ_o is split into two terms representing the two fluids.

$$B = (\rho_o - \rho_{air})\,\frac{q_o\, H_o}{H_o + H - x} + \int_{-\infty}^{\infty} u(\rho_o - \rho_w)\, dy = B_1 + B_2 \qquad (102)$$

The conservation of buoyancy flux B_2 yields

$$\frac{dB_2}{dx} = 2\,\alpha\, u_m\,(\rho_o - \rho_a) \qquad (103)$$

Eqs. (102) and (103) give

$$\frac{d}{dx}\left(u_m(\rho_o - \rho_a)\, b\,\sqrt{\pi} - u_m\,\Delta\rho_w\, b\,\frac{\sqrt{\pi}}{\sqrt{2}}\right) = 2\,\alpha\, u_m\,(\rho_o - \rho_a) \qquad (104)$$

Hence, we have to solve for u_m, c_m, $\Delta\rho_w$ and b Eqs. (94), (96), (101) and (104). These equations have to be solved numerically following the same procedure as previously outlined. In addition to the scaling parameter, P, and the source parameter, G, we will now have to consider also a stratification parameter. At some level the total driving buoyancy force of the air-bubble plume may vanish. Whether there is a complete uncoupling at such a transition level or at a higher ceiling level (overshooting) or a gradual peeling off of plume water has to be studied in the laboratory.

Detailed numerical analyses and laboratory experiments are necessary to demonstrate the validity of the theory proposed above before it may be reasonably applied to field installations.

CONCLUSIONS

Experimental observations on the velocity distribution within the air-bubble plume clearly demonstrates the close relation to a simple plume of buoyancy only. In the air-bubble case the expansion of the air due to its compressibility increases the flux of buoyancy with distance from source. The differential velocity between the rising air bubbles and the ambient water has an opposite effect as it reduces the local concentration of air bubbles in the plume. Although the two effects do not cancel each other, the simple plume equations were found to be a fairly good description of the flow field at least in the three-dimensional case. The present theory, however, makes it possible to describe with sufficient accuracy the gross behavior of any air-bubble system. More experimental data are needed to establish proper values of the coefficient of entrainment and slip velocities of the air-bubbles to cover the whole range of conditions that may apply in a practical situation. Air-bubble systems in stratified water will be subjected to further studies.

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